DOCUMENT RESUME

ED 300 230 SE 049 742

AUTHOR Rosnick, Peter
TITLE Students' Symbolization Processes in Algebra.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 82
GRANT SED-80-16567
NOTE 29p.
PUB TYPE Reports - Research/Technical (143)

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS *Algebra; Cognitive Development; *College Mathematics; *Concept Formation; Educational Research; Higher Education; Interviews; *Mathematics Instruction; Problem Solving; *Symbols (Mathematics)
IDENTIFIERS *Mathematics Education Research

ABSTRACT

The focus of this document is on students' understanding of symbolization processes in algebra, and specifically their understanding of semantically laden letters. In the first four sections, four inappropriate ways in which many college students use semantically laden letters are described. Evidence from clinical interview protocols demonstrate students' use of letters in solving algebra word problems in an imprecise, inconsistent, paradoxical, and over-associative manner. In the fifth section, a unifying theory is presented that considers each of the described student behaviors as stemming from the same conceptual framework. Students appear to view the referents of semantically laden letters as undifferentiated conglomerates. Finally, in a sixth section, implications of the data and inferences for students' understanding of symbolization processes in general, for cognitive psychologists, and for teachers and curriculum developers are discussed. (MNS)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED300230

STUDENTS' SYMBOLIZATION PROCESSES IN ALGEBRA

Peter Rosnick

Cognitive Development Project
Department of Physics and Astronomy
University of Massachusetts
Amherst, MA 01003

1982

Research reported for this paper was supported by NSF Award SED80-16567.

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

X This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

• Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

Jack Lochhead

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

SE049742

BEST COPY AVAILABLE

In almost every scientific and social scientific venture, symbolization processes play an integral, though often tacit role. The act of encoding natural language and data into a more manageable, concise notation is not only advantageous but often virtually essential for the solution of real world problems. But very little attention is paid to how that symbolization process occurs. In mathematics for example, though algebraic variables are critical components in most higher level mathematical topics, they are presented in algebra texts with little or no fanfare and are often presented in ways that are potentially misleading. When attention is paid in the texts to translation and symbolization processes (e.g., in the context of word problems), they are often presented as algorithms which, in reality, are limited in the scope and depth of their applicability. (See Rosnick, 1982.)

In this paper, I focus my attention on students' understanding of symbolization processes in Algebra and specifically on their understanding of semantically laden letters. Semantically laden letters are letters that are most often found in the solutions of word problems and are distinguished by the fact that they stand not only for numbers alone but for numbers or amounts of some qualitative entity to which they refer. For example, semantically laden letters may stand for the number of books bought or number of dollars spent, rather than just some unknown, unrelated number. These letters represent what Schwartz (1976) has called "adjectival numbers"

and reflect, in the words of Kuchemann (1981), not only "quantity" but "quality." Semantically laden letters stand at the interface between meaning in language and meaning in mathematics and are thus critical to the encoding of real-world data and relationships into a form that is more adaptable to mathematical manipulation.

The remainder of this paper is broken into six sections. In the first four I describe and document four inappropriate ways in which many college math students use semantically laden letters. Specifically, I demonstrate, using clinical interview protocols as evidence, that students use letters in the solution of algebra word problems in an imprecise, inconsistent, paradoxical, and over-associative manner. In the fifth section, I present a unifying theory that considers each of the previously described student behaviors as stemming from the same conceptual framework. Specifically, I submit that students view the referents of semantically laden letters as undifferentiated conglomerates and that such a view of referents to letters is evidenced by that imprecise, inconsistent, paradoxical, and over-associative use of these letters. Finally, in the sixth section, I discuss some of the implications these data and inferences have for students' understanding of symbolization processes in general, for cognitive psychologists, and for teachers and curriculum developers.

Students' Imprecision in Defining Semantically Laden Letters.

The word imprecise is used here to describe students who either give no definition for the letters they use in solving algebra word problems, or who give definitions that are not quantitative. Evidence for this is common. Galvin and Bell (1977) have noted the tendency of children to identify letters with nonquantitative nouns and refer to that use of letters as "fruit salad algebra". Kuchemann (1981), Matz (1979), and Davis (1975) have also all noted that tendency in children. Clement (1982a), Rosnick (1981) and Kaput (1980) have all shown that college students continue to refer to semantically laden letters nonquantitatively as labels for concrete entities. Types of problems where this "nominalist" tendency, to use Kaput's terminology, is evidenced, are problems requiring translations from one symbol system to another. Thus, students tend to write the reversed equation, $6S = P$ to represent the fact that there are six times as many students as professors at a certain university. For many of these students, the letter S is a label for the word "students" rather than a variable standing for "number of students" as it should be. (See also Clement, Lochhead and Monk, 1981.)

Evidence for imprecise letter definition is found in more traditional word problems like the following.

> A boy bought a collection of frogs and turtles. The number of frogs that he bought was three times the number of turtles that he bought. Frogs cost 3 dollars each and turtles cost 6 dollars each. He spent $60.00 altogether. How many frogs did he buy?

Of 245 college calculus students who used algebra to attempt to solve word problems like the above that were given on a written diagnostic test, 87% either gave no definition for the variables they used or defined the letters in a general, nominalist manner such as "B is books" or "T is turtles". Those students who imprecisely defined the letters they used, scored significantly worse on the diagnostic tests than those students who were more careful about defining the variables. (See Rosnick, 1982.)

Diagnostic data alone, however, does not necessarily indicate that the student is being imprecise about the meaning of the letters. After all, many expert problem solvers do not explicitly specify the meaning of the letters they use on paper. Nevertheless, those expert problem solvers, if pressed to do so, could precisely describe the letters' quantitative referents.

In clinical interviews of ten college students, nine of whom were taking a statistics course for the social sciences and nine of whom had had one or two semesters of calculus, all ten students were imprecise in defining the letters they used. In contrast with expert problem solvers however, when asked to be more specific, these students persisted in defining the letters in an over-generalized manner ("B is books" rather than "B is the number of books") and often were only able to give more quantitative definitions in retrospect, after they found solutions to the problems.

The following example gives particularly strong evidence of imprecise letter definition. Eileen had previously taken two

semesters of a rigorous calculus course and one semester of physics. At the time of this interview, she was enrolled in a statistics course. Eileen attempts to solve the word problems in a seemingly mechanical manner, deriving equations haphazardly without defining variables and terms. She correctly solves these equations but does so without, at first, recognizing the point at which she reaches an answer. She does not recognize when she is done because she does not know the precise meaning of the letters she has used.

One problem Eileen worked on was the following:

> A person went to the store and bought pecans and cashews. He bought a total of 100 nuts. The number of pounds of pecans he bought was the same as the number of pounds of cashews. 8 pecans weigh 1 pound and 12 cashews weigh 1 pound. How many pounds of pecans did he buy?

In solving this problem, Eileen constructs the correct equation, $8P + 12P = 100$. This equation is correct if P stands for the number of pounds of pecans bought. However, in arriving at that equation, Eileen says "P is pecans" and later "8 is equal to pecans" (writing 8P = pecans) and in no way indicates that she has a more precise understanding of the meaning of the letter P.

Eileen correctly solves the equation, getting $P = 5$, <u>but does not recognize it as the correct answer to the problem</u>. She plugs the five into the original equation saying that 8 times 5 is 40 pecans, and then asks the question, "how many pounds of pecans did he buy?" The following dialogue ensues.

> S: Well, he bought 40 pecans actually in number so I would take the 40 and, umm, I would take the 40 and divide it by 8 to get 5 because there are 8 nuts in every pound so, or if there is 8 nuts in 1 pound then

how many pounds would there be for 40 nuts (writes 8 nuts/1 lb = 40/x lb). So I'd go, 40 is equal to 8 nuts (writes 40 = 8x) and get x is equal to 5.

Note the introduction of the new variable x, which should have the exact same referent as P. Furthermore, note that when Eileen writes the term 8x, she says "8 nuts" instead of "8 times the number of ponds of nuts", indicating an overgeneralized, "labels" reading of the letter x. Later on, I recapped the previous interview segments.

I: You multiplied [the 5] times 8 and got 40. Then you divided 40 by 8 to get 5 again.

S: Yeah, So I probably could have stopped but I didn't recognize that that was the answer.

I: Why do you think that... you didn't recognize the answer?

S: Because probably not until I looked back did I realize that the P I was trying to find was pounds. If I realized it first then I would have said P is pounds and P is 5 then 5 pounds of pecans is what he bought. But not until you asked me what was P, what was P standing for then I realized that it had to be pounds.

Thus Eileen acknowledges that she did not know what the letter P had meant. It is interesting to note that Eileen continues to give an imprecise definition for the letter P. Though she is correct in suggesting that P has something to do with pounds of pecans, it should not stand simply for "pounds" but for "the number of pounds" that were bought.

As mentioned above, imprecise letter definition was common to the solutions of all ten students who were interviewed. The next section details another common form of incorrect usage of

letters; one that, as we shall see, is closely tied to an imprecise definition of the letters.

Students' Inconsistent Use of Semantically Laden Letters.

Students most frequently demonstrate inconsistent letter usage by shifting between related but different meanings of the same letter. One student, for example, was working on a problem using the letter B which should have meant "the number of books bought". In the course of working on this single problem she made all of the following inconsistent statements.

"...Use B for books."

"B is the number of books which is 2 dollars each."

"B is the number of books, but it-, more importantly...B is -is a price."

"B is one book."

"B [is] the number of books."

[B stands for the total amount of money spent on books (i.e., price times quantity).]

What is remarkable is that each consecutive definition of the letter is made without rejecting the previous definition. There is a sense that for this student, each statement is mutually compatible. Though few of the ten students shifted the meaning of the letters they used as blatantly and frequently as the above student, nine of the ten were inconsistent in their use of letters. I must emphazise that the shifts that did occur seemed to be unconscious. In other words, there was no indication that the students were consciously "changing their

minds" as they moved from one interpretation of the letter to another.

The following example shows how being imprecise about defining letters can lead to inconsistent usage. Paul, a student enrolled in a statistics course who had had one semester of calculus, is working on the following problem.

> A person went shopping for books and records. He spent a total of $72. The price of each book was the same as the price of each record. He bought two books and six records. What is the price of one record?

Like Eileen, Paul quickly produces a correct equation, solves it, but does not recognize that his answer is the solution to the problem.

Paul's equation, $72 = 2x + 6x$, is correct if x is defined as the price of a book or record. (Those prices are given as being the same.) However, Paul reads the equation in the following way.

S: $72 equals 2 items plus 6 items.

For Paul, x is identified with the label, "items", rather than the appropriate definition, "the cost of an item". When he solves the equation, getting [x = 9], (which is the answer to the problem), he becomes confused because he is unclear about what x means. The following dialogue ensued.?

S: It [x = 9] doesn't seem to be answering the question.

I: How's that?

S: Uh---

I: When you say 9 is equal to x, what do you conclude from that?

S: I don't know.

I: ...What does x mean?

S: A minute ago I said x means items purchased.

I: Uh-huh.

S: 9 items purchased.

I: Uh-huh. Does that seem reasonable to you?

S: Yeah.

I: Ok.

S: ---(30 seconds)---No.

I: What was that?

S: I said no, it doesn't seem reasonable.

I: Oh.

S: 8 items purchased.

I: Uh-huh. You pointed to the 8x when you said that.

S: Yeah. The x is items purchased. This is all very confusing.

I: Uh-huh. It's a confusing problem.

S: So I guess uh,--it's $9 for 1 record.

I: How did you decide that? Now what you just did was, you put dollar signs on the 9 and the 72 and you just said $9 for each record. How did you come to that conclusion?

S: There was 8 items purchased.

I: Uh-huh.

S: For a total of $72. And the price of each book was the same as the price of each record. So everything costs $9.

I: I see. Um, so you say 8 items were purchased when you look at which equation?

S: $72 = 8x; 8 items.

I: Uh-huh. So x means what?

S: Items purchased.

At this point, it seems that, to Paul, the x still means "items purchased" despite the fact that he has earlier written x = $9, indicating that x stands for the cost of an item.

I: Items purchased. And then you have x is equal to $9.

S: Right. Items purchased equals $9; those were the prices. $9 per item purchased.

I: It sounds like you have a question in your voice when you're saying that.

S: Uh, each item costs $9.

Up until this point, Paul had not been able to resolve the apparent contradiction that 8x meant that there were 8 "items purchased" but that x = $9. The x seemed to play a dual and shifting role for Paul.

I: Uh-huh; So say again now specifically what x means.

S: The cost for an item.

I: The cost for an item. Did you, er, were you thinking that when you first wrote the equation down? You just said x is the cost of 1 item. Were you thinking that when you first wrote it down?

S: I don't think all that consciously, but maybe a little bit subconsciously. I think that's what it had to be; yeah.

If, in fact, Paul was thinking correctly subconsciously, it must have been quite far back in his consciousness because he didn't recognize the right answer when he saw it. A more plausible explanation is that because the qualitative concept "items purchased" was mixed with the quantitative attribute, "price of an item", Paul became confused.

Note that in both examples of inconsistent letter usage, the shifts seem to occur over time. In the next section, I present evidence of students using contradictory meanings of a letter simultaneously.

A Paradoxical Use of Semantically Laden Letters

In the previous section, I claimed that students are often not concious of the shifts in meaning of the letters they use. This claim is supported by evidence of what I will call a paradoxical use of letters. Students use letters paradoxically when they accept the juxtaposition of two contradictory quantitative uses of the same letter. They may, for example, write two equations that are both true only if the letters used in the equations are given different meanings. But again, the students seem not to be conscious of the contradictory usage of the letters. They may, in fact, try to solve those mutually incompatible equations simultaneously.

As was the case with imprecise letter definition, this behavior seems to occur frequently on written diagnostic tests. One student, for example, wrote the following series of contradictory equations on his test paper.

$$r + g = 91$$

$$1r + 3g = 91$$

$$1r + 4(3g) = 91$$

However, it is unclear from the written data alone whether each successive equation implies a refinement and rejection of the one

before, or, on the other hand, whether there is a belief that all three equations are correct but only the last one is needed to solve the problem.

The clinical interview data seem to imply that the latter is quite often the more accurate discription. Consider the following example. Ray is also enrolled in a statistics course and also has taken a course in calculus. He is working on the same pecans and cashews problem that Eileen worked on. After he reads the problem, he writes down the information using a labels approach. Thus, he writes 8P = 1 lb and 12C = 1 lb indicating that there are eight pecans and, respectively, twelve cashews to a pound. He quickly follows this by writing P = C, saying "the number of pounds of pecans he bought was the same as the number of pounds of cashews". Note the inconsistency. P first means the word "pecans" and then means the "number of pecans bought". Ray becomes stuck in trying to solve the problem algebraically and resorts to a trial and error, arithmetic solution without using variables. He is successful after several attempts. We pick up the transcript at that point.

I: So you solved it by trial and error.

S: Right. I just plugged in a couple of numbers. I'm sure I could have worked it out in a formula.

I: Could you do that for me? I'd be interested in seeing the formula.

S: You want a formula. Well, you know P equals C, so...So it's got to be, umm 12C + 8P = 100 and let me see. P equals 3/4C because 8 is...No, 2/3, I'm sorry (writes P = 2/3C). 8 is two thirds of 12, right.

Note the paradox. Both equations, 12C + 8P = 100 and P = 2/3C

are correct, but only if the letters have different quantitative meanings. In the former, P for example, has to stand for the number of pounds of pecans bought. In the latter, P has to stand for either the number 8 or the number of pecans bought. The interview continues.

I: So P is equal to two thirds C.

S: So when you multiply...No, it doesn't work either.

I: what were you about to do?

S: I was going to multiply 8 times two thirds but it comes out to 16, it comes out to 16 thirds and that's not a very nice figure to work with.

Ray is trying to solve the two equations simultaneously but rejects that, not because the letters mean different things in each case, but merely because the numbers come out to be non integral. Later, I ask:

I: What does the letter C [in 12C + 8P = 100] stand for?

S: Cashews. And P stands for pecans. But they're worried about weight, how many pounds of pecans did he buy. You know it's equal to the number of pounds of cashews he bought so, 8 pecans weigh 1 pound and 12 cashews weigh 1 pound. So, you know there is 100 nuts. Let me check this over again. I want to make sure this proportion is right. 12 cashews in a pound and 8 pecans in a pound so 8 over 12 it's got to be two thirds.

Despite acknowledging that the problem is "worried about weight", Ray does not let go of trying to relate P and C in terms of the numbers of individual pecans and cashews, respectively, in a pound.

Eventually, Ray gives up on the equation, P = 2/3C, only to try the semantically identical equation, 3/2P = C. When that

also produces a non integral result, he goes back to rereading the problem. He rediscovers that he can write P = C and solves the problem accordingly. What is important to note is that he never completely rejects the idea of relating P with C via the equation P = 2/3C. I ask:

I: What do you think was wrong with what you were doing over here when you said P was equal to two thirds C?

S: Well, I wasn't, I wasn't, umm...I was trying to solve..I wasn't...

S: ...What I was doing over here, I was trying to make C and P equal in a proportional sense, you know, by using a fraction, by dividing the 8 into 12 or the 12 over the 8, when...

I: What's wrong with that?

S: Well, it's not necessary because it's given, it's C equals P...

Thus it seems that for Ray, there is nothing wrong or inconsistent with writing P = 2/3C, it is just that it is not necessary in solving the problem. Ray does not seem to recognize that the meaning of the P is different in P = 2/3C from its meaning in 8P + 12C = 100. This apparent acceptance of the juxtaposition of two contradictory quantitative uses of the same letter exemplifies what I mean by paradoxical usage.

Students' Over-associative Use of Semantically Laden Letters.

Time and time again in the clinical interviews, when I asked students to define what the letters meant, their answers were over-generalized and nominalistic. Eileen said that she would "use P for Pecans" and, later, that P is "pounds"; Paul said x

means "items purchased"; and Ray said "P stands for pecans." This form of imprecise letter definition seems to indicate that the letter is being over-associated with the qualitative component of the referent of that letter. This over-association is most clearly evidenced in the clinical interviews when a student is incapable of dropping the semantic load of a letter, even when it is appropriate to do so. For example, in the following problem , it is given that "the number of books bought" was the same as "the number of records" bought.

> I went to the store and bought the same number of books as records. Books cost two dollars each and records cost six dollars each. I spent $40 altogether. Assuming that the equation 2B + 6R = 40 is correct, what is wrong, if anything, with the following reasoning. Be as detailed as possible.
>
> 2B + 6R = 40 Since B = R, I can write
> 2B + 6B = 40
> 8B = 40
>
> This last equation says 8 books is equal to $40. So one book costs $5.

Thus, it is perfectly legitimate to write B = R when B and R are the numbers of books and records bought, respectively. However, a common, incorrect response to this problem is that B can't equal R because "books" aren't the same things as "records."

(This problem was originally given to thirty calculus students as a written test. The results show that 23 students (77%) were unable to recognize the misinterpretation of the last line. That is, they did not recognize that the line 8B = 40 reads as "8 times the number of books bought is equal to 40 dollars," rather than "8 books equals 40 dollars.")

In a clinical interview, a calculus student, Beth, is

working on this problem. She is very troubled by the equation 2B + 6B = 40, which appears in the text of the problem, because, as she says, "why would you multiply the cost of each record [$6] times the amount of books [B]." Even though the amount of books equals the amount of records "B is not equal to R because um, you cannot substitute B for R or R for B...and have everything else be right." It appears that she cannot let go of the fact that the referent of B is "books" and thus B should in no way be related to any attribute of records, specifically their 6 dollar cost. B has more meaning than just its quantitative value. Eventually Beth correctly recognizes that 5 books and 5 records were bought. But she is still convinced that the algebraic solution that was given to her is incorrect. She is still troubled by the equation, B = R.

S: ...B does not equal R because B is a book and R is a record so a book doesn't equal a record.

I: Okay. B is a book and R is a record?

S: Right. A-an amount...an unspecified amount...it could be zero, it could be 50 million, but in this case, 5 happens to be an answer.

I: Okay. But just because both of them are 5, they're still not equal?

S: Right. Because books aren't equal to...Not if-it's just like saying up..to-you know, tomato is to an orange.

I: Uh-huh, uh-huh...uh-huh.

S: You know, the amounts can happen to be the same but the just the amount of-of the book ...I don't know. I'm getting myself in deeper with this...(9 secs)..the amounts here are the same but they're not the amounts of the same thing.

Thus, for Beth, the over-association of the letters with the semantic components of the objects they refer to hinders her ability to submit the equations to standard rules of algebra. Specifically, she is unable to replace one letter with its numeric equivalent which would lead to a solution of the system of equations.

Students' Conceptual Orientation Concerning Semantically Laden Letters.

The previous four sections describe observable behaviors of students who are working with semantically laden letters. In this section, I propose an explanation of what might be going on internally for these students. That is, I propose an hypothesis for describing how students conceptualize semantically laden letters which causes them to behave in the above ways.

To do so, I must first resolve an apparent anomoly in the clinical evidence. This anomoly is most vividly embodied in students' paradoxical usage of letters but is also evident in their using letters inconsistently and over-associatively. The evidence suggests that students are not differentiating between attributes of semantic entities, like price and quantity. Yet, surely, college students know that the price of an item is different from the quantity purchased. Why don't they acknowledge that difference when using semantically laden letters?

The answer may be found in a closer analysis of the fourth behavior, the over-association of the letter with its complex

referent. We have seen where that over-association prevented students from replacing a letter with its numeric equivalent because to do so would be like comparing "tomatoes" with "oranges." It seems, however, that that over-association goes beyond an over-identification of the letter with the qualitative component of the referent. In the "Books and Records" problem discussed above, the most common answer given in the written tests for why B cannot equal R is that, "even though the numbers are the same, B can't equal R because their prices are different." It seems as though the students are not only over-associating the letter with the qualitative component of the complex referent, but also with multiple quantitative attributes of that referent. B seems not only to be standing for "books" in the students' minds, and not only for quantity, but also for the price of a book, and possibly other attributes as well.

Thus, perhaps students' perplexing behaviors can be explained as follows. When students use letters in, what appear to us to be, inconsistent and paradoxical manners, perhaps the apparent shifts and contradictions are for the student only different manifestations of one broad, undifferentiated concept. This concept might be an amorphous congeries of meanings including, for example, the label "books", the quantitative attributes "quantity of books," "price per book," and "total price of all of the books" and, etc.

Put differently, the student may associate the letter, B, with an undifferentiated concept called "books" or if you will,

"bookness", and thus B would stand for everything that the words "book" or "books" imply. That B is used to stand both for price and quantity as well as a nominalist label is a logical outcome of such a concept.

The hypothesis that describes students' conceptual orientation from which their misuse of semantically laden letters evolves can now be restated as follows. For many students, the mental referent of a semantically laden letter is an undifferentiated conglomerate. They tend to identify letters used in word problem solutions with entire, complex, overly generalized referents rather than particular quantitative attributes of those referents. The outcome of such a view of letters is the imprecise, inconsistent, paradoxical, and over-associative usage described in the four preceeding sections.

Had it been the case that each of the four behaviors appeared in isolated, unconnected situations, one might be inclined to explain them away as just carelessness on the part of the students. One might, consequently, be less inclined to find a unifying explanation for why students use semantically laden letters as they do. However, these behaviors are not isolated and unconnected, and a unifying hypothesis, like the above, helps explain these co-occurrences.

In fact, I have concluded that nine of the ten students who were interviewed demonstrated at least three of the four behaviors while working on a problem. They repeatedly allowed the meaning of the letters to shift, or used the letters

paradoxically, or associated the letter with the qualitative component of the complex referent, all the while defining the letters in an imprecise and overgeneralized manner.

This co-occurrence of the behaviors is explained and might have been predicted by the hypothesis that has been set forth in this section. In the next section, I will describe some of the implications of these findings which can be broken into three categories: implications for students' understanding of symbolization processes in general; implications for cognitive psychology; and implications for teachers and curriculum developers.

Implications

Students' Understanding of Symbolization Processes. As I mentioned in the introduction, symbols and symbolization processes play an integral if not essential role in the application and advancement of most scientific and social scientific fields. In this section, I submit that the student difficulties described above concerning their understanding of algebraic variables may reflect a broader difficulty they have with symbolization processes in general. This in turn suggests that any difficulty students have with word problem solving in various topics and fields could, in part, be explained by their inability to fully understand the symbols they must use to solve those problems.

To illustrate, consider the "maximum/minimum" problems that

are found in most calculus courses. Most teachers of calculus would agree that these problems are among the most difficult to teach. Certainly, a major part of the problem is that students do not sufficiently understand the concept of a derivative. Misconceptions concerning functions and the continuous nature of the variables are also likely contributors to confusion. (See Kaput, 1981.) However, it seems likely that the students are further handicapped by the fact that these problems require the use of semantically laden letters. In many problems, like the proverbial "fence" problems (what is the largest area given a fixed perimeter, or what is the smallest perimeter, thus cost of materials, given a fixed area), there is more than one relevant attribute to consider (perimeter versus area, or quantity versus cost, etc.). It is reasonable to expect that the students' misuse and misunderstanding of semantically laden letters would be a significant contributor to their difficulties with the maximum/minimum problems.

Similarly, the fact that the concept of a random variable in statistics is a particularly difficult one for students to understand might be related to their confused notions about variables in algebra. In fact, in most mathematical domains where variables are used, especially where they are semantically laden, it is reasonable to expect that the misconceptions described above will greatly affect their ability to solve problems in that domain.

The same may be true for domains outside of mathematics

where symbolization processes occur. In physics for example, Reif (1977) has found that students are often incapable of

> making appropriate discriminations between symbols (either quantities or entire relations) and their referents...so that a particular symbol is correctly assigned to a particular referent, without being either misassigned to inappropriate referents or confused with other symbols. (p.5)

Similarly, Soloway et al (1981) report that 28% of students who are learning a computer language wrote programs "in which the same variable was used incorrectly for more than 1 role." They discribe this multiple use of the variables as "mushed variables," a term which colloquially encapsulates much of what I mean by "undifferentiated conglomerates."

The list of domains involving some form of semantically laden symbolization processes would be a very long one. In Chemistry, Business, Economics, Psychology, and any other field that requires some facility with symbolized quantities, students' ability to solve problems in those fields is likely to be negatively affected by their misconceptions concerning semantically laden letters.

Bamberger and Schon (1978) have shown that even in music, subjects learning for the first time to symbolically encode musical notes are often imprecise and inconsistent in labeling notes with symbols. One young subject, for example, used a single written symbol to stand alternatively (and in a contradictory way) for a bell, a beat, and a note. Bamberger and Schon also say that in

> every domain of human experience where the interaction of apprehended sensory figures and symbol systems is

> possible --for example, mathematics--... in order for figures to enter into a symbol system, work must be done... Situationally dependent figures must be made into named, differentiated elements and relations which can be held constant over time and context and person.

Until students learn to do that "work", their ability to use symbol systems, and thus their ability to solve problems, will be greatly limited.

Implications for Cognitive Psychology. In the section beginning on page 18, I hypothesized that the students' conceptual framework that is responsible for their imprecise, inconsistent, paradoxical, and over-associative use of letters is that they view the referent of a semantically laden letter as an undifferentiated conglomerate. They do not seem to differentiate between various quantitative and qualitative attributes of a referent. An interesting open question is whether or not this conceptual framework is common to preliminary phases of concept formation in other areas. That is, are their other areas where people tend to inadvisedly clump together related but distinct concepts into an undifferentiated mass?

Another example drawn from Physics suggests that this might be the case. Reif, (1982) has noted that students have difficulty in differentiating between different forms of energy, i.e. kinetic, potential, human (as in vitality), etc. Apparently, the term energy is used in an overly generalized fashion whereby students do not discriminate as to what type of energy they are talking about. It might be the case that for these students, the word "energy" refers to an "undifferentiated

conglomerate" which amorphously subsumes all uses and meanings of the word.

It seems likely that with many concepts like energy in physics and the symbolization processs in Algebra, one phase of people's conceptual development is to go through a period of time when they view the concept without precision, consistency, and discrimination. It also seems likely that with many such concepts, some people never get beyond that phase. Thus, the concept of an undifferentiated conglomerate may prove to be a very important one for any theory of conceptual development.

Implications for Teachers and Curriculum Developers. I have provided evidence suggesting that many students use letters in an imprecise, inconsistent, paradoxical, and over-associative manner and have hypothesized that these students view the referents of these letters as undifferentiated conglomerates. Many students are thus coming from two years or more of high school and precalculus algebra without developing an adequate understanding of the use and meaning of the variables that dominate their course work. Clement (1982b) put it best by saying that though students may be adept at symbol manipulation processes, they have a disturbing lack of adeptness with symbol generation and symbol interpretation processes. The latter two processes are essential to the application of mathematics to real world contexts.

These findings have immediate implications for teachers and curriculum developers. Simply put, teachers of all subjects should be cognizant of the fact that their students may have

difficulty with all aspects of symbolization processes, and should not be content with a curriculum that focuses primarily on symbol manipulation processes. Much more attention should be paid to how variables are generated and interpreted in problem solving contexts.

Teachers of all subjects should be aware that a student's difficulty with problem solving might be a reflection of their poorly developed understanding of how symbols are used. For example, Economics students may have as much difficulty understanding symbols used to stand for Supply and Demand as they have understanding the economic principles behind Supply and Demand.

Furthermore, teachers and curriculum developers should also pay attention to those concepts (like possibly the derivative in calculus) for which it is likely that students will become stuck in a phase when they view that concept as an undifferentiated conglomerate. Developing strategies that pull students through these phases of imprecise, undifferentiated thinking may be no easy task, but nevertheless, an essential one. For if students have not developed the appropriate concepts and adequate thinking skills, they will constantly run into difficulties in solving problems, in school, and out.

REFERENCES

Bamberger, Jeanne and Schon, Donald. "The Figural-Formal Transaction." Working Paper. Massachusetts Institute of Technology, Cambridge, 1978.

Clement, John. "Algebra Word Problem Solutions: Analysis of a Common Misconception." Journal for Research in Mathematics Education 13 (January 1982a): 16-30.

Clement, John. Personal Communication, 1982b.

Clement, John, Lochhead, John and Monk, G. "Translation Difficulties in Learning Mathematics." American Mathematical Monthly 88 (April 1981): 286-290.

Davis, R. "Cognitive Processes Involved in Solving Simple Algebraic Equations." The Journal of Children's Mathematical Behavior 1 (Summer 1975): 7-35.

Galvin, W.P. and Bell, A.W. "Aspects of Difficulties in the Solution of Problems Involving The Formulation of Equations." Nottingham Skill Centre for Mathematical Education, University of Nottingham, 1977.

Kaput, James. "Comparing Formal Algebraic Representations and Natural Representational Processes." Proposal submitted to National Science Foundation RISE Program, 1980.

Kuchemann, Dietmar. "Children's Understanding of Numerical Variables." Mathematics in School 7 (September 1978): 23-26.

_____. "Object Lesson In Algebra?" Technical report, University of London Institute of Education, July, 1981.

Matz, Marilyn. "Towards a Process Model for High School Algebra Errors." Working paper 181, Artificial Intelligence Laboratory, Massachusetts Institute of Technology, Cambridge, 1979.

Reif, Frederick. "Problem Solving in Physics or Engineering: Human Information Processing and Some Teaching Suggestions." In The Teaching of Elementary Problem Solving in Engineering and Related Fields pp. 179-198. Edited by James Lubkin. Washington: American Society for Engineering Education, 1982.

_____. Personal communication, University of California, Berkeley, 1982.

Rosnick, Peter. "Some Misconceptions Concerning the Concept of

Variable." The Mathematics Teacher 74 (September 1981): 418-420.

_____. "The Use of Letters in Precalculus Algebra." Ed.D dissertation, University of Massachusetts, 1982.

Schwartz, Judah. "The Semantic Aspects of Quantity." Technical report, Massachusetts Institute of Technology, Cambridge, May, 1976.

Soloway, Elliot, Ehrlich, Kate, Bonar, Jeffrey, and Greenspan,Judith. "What do Novices Know About Programming?" To appear in Directions in Human-Computer Interaction. B. Shneiderman and A. Badre, Editors.

Tonnessen, L. "Measurement of the Levels of Attainment by College Mathematics Students of the Concept Variable." Ph.D. dissertation, University of Wisconsin, Madison, 1980.